Bonus Malus Systems in Car Insurance

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Car Insurance Variables

- A priori classification variables: age, gender, type and use of car, country
- A posteriori variables: deductibles, credibility, bonus-malus

Bonus-Malus:

- Answer to **heterogeneity of behavior** of drivers
- Inducement to drive more carefully
- Strongly influenced by regulatory environment and culture

Are you a safe driver?

- Yes
- No

Who are safer drivers?

- Women
- Men

March 2011

Statistically, men drive more recklessly and cause more severe accidents than women. Which is why men tend to pay more than women to insure their cars.

European Court of Justice agreed: for all the damning evidence of men behaving badly, **gender can no longer play a part** in how much someone pays for insurance.

From "E.U. Court to Insurers: Stop Making Men Pay More", By Leo Cendrowicz, TIME, Mar. 02, 2011

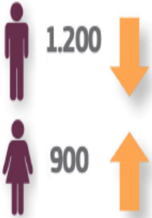
December 2012

EU rules on gender-neutral pricing in insurance.

From 21 December 2012, insurance companies in the European Union will have to charge the same price to men and women for the same insurance products, without distinction on the grounds of gender.

Example 1:

John and Mary are both 18 years old and drive the same type of car. John currently pays 1200 euros a year for car insurance and Mary 900. Under the new rules, John and Mary will pay the same premium, which should therefore increase for Mary and decrease for John.



Laurianne Krid, policy manager at FIA

"Women are safer drivers statistically, but they should pay according to their real risk, which can be calculated objectively."

"We want insurance to be based on criteria like type of vehicle, the age of the driver, how much you drive during the year, and **how many accidents you have had**."

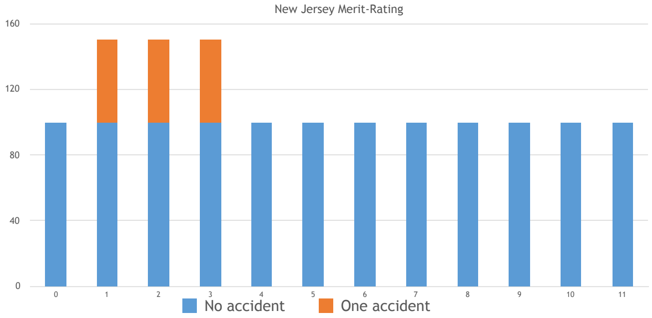
From "E.U. Court to Insurers: Stop Making Men Pay More", By Leo Cendrowicz, TIME, Mar. 02, 2011

Bonus-Malus System (BMS)

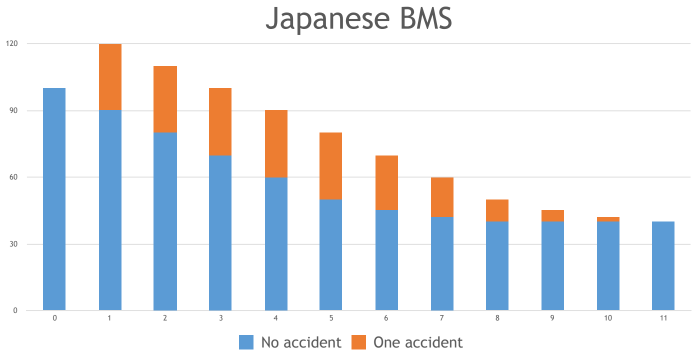
- Merit Rating System
- Fair Premium sharing
- No Claim Discount

Note: Bonus-Hunger Problem

Example - New Jersey BMS



Example - Japan BMS



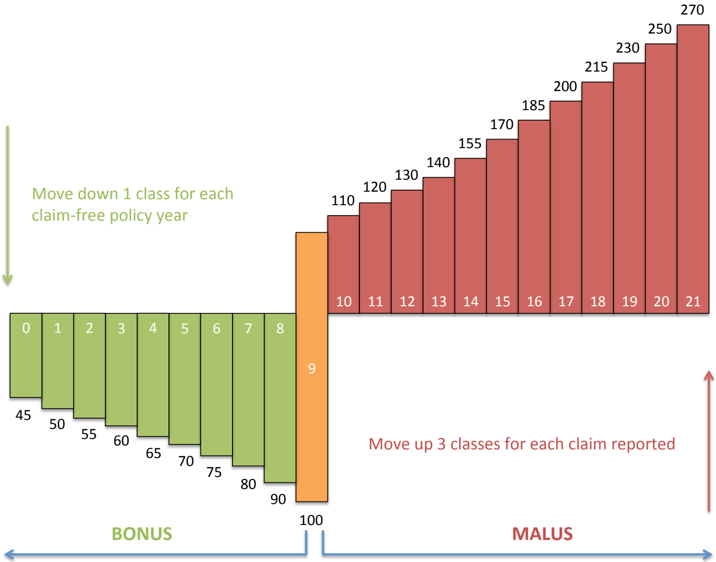
Japan Bonus-Malus System

Japanese BMS

Class	Premium	Class after x claims					
		0	1	2	3	4	5
16	150	15	16	16	16	16	16
15	140	14	16	16	16	16	16
14	130	13	16	16	16	16	16
13	120	12	16	16	16	16	16
12	110	11	15	16	16	16	16
11	100	10	14	16	16	16	16
10	90	9	13	16	16	16	16
9	80	8	12	15	16	16	16
8	70	7	11	14	16	16	16
7	60	6	10	13	16	16	16
6	50	5	9	12	15	16	16
5	45	4	8	11	14	16	16
4	42	3	7	10	13	16	16
3	40	2	6	9	12	15	16
2	40	1	5	8	11	14	16
1	40	1	4	7	10	13	16

Example - Switzerland BMS

A Swiss Bonus-Malus system with premium levels as percentages to the base premium



Swiss Bonus-Malus Scale (Dufresne, 1988)

Premiums as a percentage of the base premium

x	%	x	%	x	%
0	45	8	90	16	185
1	50	9	100	17	200
2	55	10	110	18	215
3	60	11	120	19	230
4	65	12	130	20	250
5	70	13	140	21	270
6	75	14	155		
7	80	15	170		

Lemaire (1995)

- optimal Bonus-Malus Systems (BMS) -assign to each policyholder a premium based only on the number of his accidents.
- same penalty for an accident of a small size or big size.
- optimality is obtained by minimizing the insurers risk.

$$NetPremium = E(Frequency)\underbrace{E(Severity)}_{constant}$$

Example of number of claims N

On a third party liability insurance:

Number of claims	Observed policies
0	96, 978
1	9, 240
2	704
3	43
4	9
5+	0
Total	106, 974

$Mean = 0.1011$ and $Variance = 0.1070$

Poisson fit for number of claims N

$$P(N = n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

- $E(N) = Var(N) = \lambda$
- The non-contagion distribution: independent increments
- Stationary increments
- MLE and MM lead to the same estimator of λ, $\hat{\lambda} = 0.1011$

Same example with Poisson fit for number of claims N

On a third party liability insurance:

Number of claims	Observed policies	Poisson fit
0	96,978	96,689.6
1	9,240	9,773.5
2	704	493.9
3	43	16.6
4	9	0.4
5+	0	0
Total	106,974	106,974

Mixed Poisson distributions

- Obviously Poisson is not the best fit!
- Need a distribution that exhibits positive contagion (dependence)
- Still assume that each individual has claims according to a Poisson(λ) process
- However, assume λ is a continuous random variable with density $g(\lambda)$,

$$P(N(t) = n) = \int_0^\infty P(N(t) = \lambda \mid \lambda) g(\lambda) d\lambda$$

Negative Binomial distribution N

For $g(\lambda)$ we can select the *Gamma*(m, θ) distribution

$$g(\lambda) = \frac{\theta^m \lambda^{m-1} e^{-\theta\lambda}}{\Gamma(m)}$$

Then $N(t)$ follows a *Negative Binomial* (m, θ) distribution

$$P(N(t) = n) = \frac{\theta^m t^n}{(t+\theta)^{n+m}} \frac{\Gamma(n+m)}{n!\Gamma(m)}$$

with $E(N(t)) = \frac{m}{\theta} t$ and $Var(N(t)) = \frac{m}{\theta} t + \frac{m}{\theta^2} t^2 > E(N(t))$

Modelling Negative Binomial Claim Frequency

Mixing Poisson with Gamma results in Negative Binomial

$$P(N = n) = \int_0^\infty \frac{\mathrm{e}^{-\lambda}\lambda^n}{n!} \cdot \frac{\lambda^{m-1}\theta^m \mathrm{e}^{-\theta\lambda}}{\Gamma(m)} \mathrm{d}\lambda = \binom{n+m-1}{n} \theta^m \left(\frac{1}{1+\theta}\right)^{m+n}.$$

Bayesian Approach - Posterior Distribution $Gamma(K + m, t + \theta)$

$$\mu(\lambda | k_1, k_2, \dots, k_t) = \frac{(\theta + t)^{K+m} \lambda^{K+m-1} \mathrm{e}^{-(t+\theta)\lambda}}{\Gamma(m + K)}, \quad K = \sum_{i=1}^{t} k_i$$

Best Estimate (quadratic loss function): posterior mean

$$E[Frequency] = \lambda_{t+1}(k_1, k_2, \dots, k_t) = \frac{m + K}{t + \theta}.$$

Same example with NB fit for number of claims N

On a third party liability insurance:

Number of claims	Observed policies	Poisson fit	NB fit
0	96, 978	96, 689.6	96, 985.5
1	9, 240	9, 773.5	9, 222.5
2	704	493.9	711.7
3	43	16.6	50.7
4	9	0.4	3.6
5+	0	0	0
Total	106, 974	106, 974	106, 974

Note: MM to estimate $\hat{m} = 1.6049$ and $\hat{\theta} = 15.8778$.

Average number of claims

- Apriori - $Gamma(m, \theta)$: $\hat{\lambda} = \frac{m}{\theta}$
- Observe claim history: $\{k_1, k_2, \ldots, k_t\}$, $k = k_1 + \cdots + k_t$
- Aposteiori - $Gamma(m + k, \theta + t)$: $\hat{\lambda} = \frac{m+k}{\theta+t}$

Net Premium in Optimal BMS

$$NetPremium = \underbrace{E(Frequency)}_{=\hat{\lambda}=\frac{m+k}{\theta+t}} * \underbrace{E(Severity)}_{=constant}$$

Examples:

- Time 0: $P_1 = \frac{m}{\theta} = 0.1011$ Set as 100.
- Time 1:
  - $k_1 = 0$: $P_2 = \frac{m}{\theta+1} = 0.0951$ Set 94.
  - $k_1 = 1$: $P_2 = \frac{m+1}{\theta+1} = 0.1543$ Set 153.
- Time 2:
  - $k_1 = 0, k_2 = 0$: $P_3 = \frac{m}{\theta+2} = 0.0898$ Set 89.
  - $k_1 = 1, k_2 = 3$: $P_3 = \frac{m+4}{\theta+2} = 0.3135$ Set 310.

Net Premium in Optimal BMS

Optimal BMS with Negative Binomial model

Year	Claims				
	0	1	2	3	4
0	100				
1	94	153	211	269	329
2	89	144	199	255	310
3	84	137	189	241	294
4	80	130	179	229	279
5	76	123	171	218	266
6	73	118	163	208	253
7	69	113	156	199	242

Optimal BMS with NB

Advantages:

- FAIR - as a result of Bayes rule
- Financially balanced - the average income of the insurer remains 100 every year

Disadvantages:

- high penalties
- encourages uninsured driving
- suggests hit and run behaviour
- induces the policyholder to change the company after one accident.

Instead Markov chains are used in practice.

Japan Bonus-Malus System (Lemaire, 2017)

Japanese BMS

Class	Premium	Class after x claims					
		0	1	2	3	4	5
16	150	15	16	16	16	16	16
15	140	14	16	16	16	16	16
14	130	13	16	16	16	16	16
13	120	12	16	16	16	16	16
12	110	11	15	16	16	16	16
11	100	10	14	16	16	16	16
10	90	9	13	16	16	16	16
9	80	8	12	15	16	16	16
8	70	7	11	14	16	16	16
7	60	6	10	13	16	16	16
6	50	5	9	12	15	16	16
5	45	4	8	11	14	16	16
4	42	3	7	10	13	16	16
3	40	2	6	9	12	15	16
2	40	1	5	8	11	14	16
1	40	1	4	7	10	13	16

Japan BMS transition matrix

Japanese BMS transition matrix

	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
16	$1-p_0$	p_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	$1-p_0$	0	p_0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	$1-p_0$	0	0	p_0	0	0	0	0	0	0	0	0	0	0	0	0
13	$1-p_0$	0	0	0	p_0	0	0	0	0	0	0	0	0	0	0	0
12	$1-p_0-p_1$	p_1	0	0	0	p_0	0	0	0	0	0	0	0	0	0	0
11	$1-p_0-p_1$	0	p_1	0	0	0	p_0	0	0	0	0	0	0	0	0	0
10	$1-p_0-p_1$	0	0	p_1	0	0	0	p_0	0	0	0	0	0	0	0	0
9	$1-p_0-p_1-p_2$	p_2	0	0	p_1	0	0	0	p_0	0	0	0	0	0	0	0
8	$1-p_0-p_1-p_2$	0	p_2	0	0	p_1	0	0	0	P_0	0	0	0	0	0	0
7	$1-p_0-p_1-p_2$	0	0	p_2	0	0	p_1	0	0	0	p_0	0	0	0	0	0
6	$1-p_0-p_1-p_2-p_3$	p_3	0	0	p_2	0	0	p_1	0	0	0	p_0	0	0	0	0
5	$1-p_0-p_1-p_2-p_3$	0	p_3	0	0	p_2	0	0	P_1	0	0	0	p_0	0	0	0
4	$1-p_0-p_1-p_2-p_3$	0	0	p_3	0	0	p_2	0	0	p_1	0	0	0	p_0	0	0
3	$1-p_0-p_1-p_2-p_3-p_4$	p_4	0	0	p_3	0	0	p_2	0	0	p_1	0	0	0	p_0	0
2	$1-p_0-p_1-p_2-p_3-p_4$	0	p_4	0	0	p_3	0	0	P_2	0	0	p_1	0	0	0	p_0
1	$1-p_0-p_1-p_2-p_3-p_4$	0	0	p_4	0	0	p_3	0	0	p_2	0	0	p_1	0	0	p_0

Japan BMS 1 step transition matrix

Japanese BMS transition matrix

	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
16	.0952	.9048	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	.0952	0	.9048	0	0	0	0	0	0	0	0	0	0	0	0	0
14	.0952	0	0	.9048	0	0	0	0	0	0	0	0	0	0	0	0
13	.0952	0	0	0	.9048	0	0	0	0	0	0	0	0	0	0	0
12	.0047	.0905	0	0	0	.9048	0	0	0	0	0	0	0	0	0	0
11	.0047	0	.0905	0	0	0	.9048	0	0	0	0	0	0	0	0	0
10	.0047	0	0	.0905	0	0	0	.9048	0	0	0	0	0	0	0	0
9	.0002	.0045	0	0	.0905		0	0	.9048	0	0	0	0	0	0	0
8	.0002	0	.0045	0	0	.0905		0	0	.9048	0	0	0	0	0	0
7	.0002	0	0	.0045	0	0	.0905	0	0	0	.9048	0	0	0	0	0
6	0	.0002	0	0	.0045	0	0	.0905	0	0	0	.9048	0	0	0	0
5	0	0	.0002	0	0	.0045	0	0	.0905	0	0	0	.9048	0	0	0
4	0	0	0	.0002	0	0	.0045	0	0	.0905	0	0	0	.9048	0	0
3	0	0	0	0	.0002	0	0	.0045	0	0	.0905	0	0	0	.9048	0
2	0	0	0	0	0	.0002	0	0	.0045	0	0	.0905	0	0	0	.9048
1	0	0	0	0	0	0	.0002	0	0	.0045	0	0	.0905	0	0	.9048

Japan BMS 4 steps transition matrix

Japanese four-step transition matrix

	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
16	.0952	.0861	.0779	.0705	.6703	0	0	0	0	0	0	0	0	0	0	0
15	.0281	.1531	.0779	.0705	0	.6703	0	0	0	0	0	0	0	0	0	0
14	.0281	.0191	.2120	.0705	0	0	.6703	0	0	0	0	0	0	0	0	0
13	.0281	.0191	.0109	.2716	0	0	0	.6703	0	0	0	0	0	0	0	0
12	.0248	.0224	.0109	.0035	.2681	0	0	0	.6703	0	0	0	0	0	0	0
11	.0047	.0358	.0176	.0035	0	.2681	0	0	0	.6703	0	0	0	0	0	0
10	.0047	.0023	.0410	.0135	0	0	.2681	0	0	0	.6703	0	0	0	0	0
9	.0046	.0024	.0008	.0403	.0134	0	0	.2681	0	0	0	.6703	0	0	0	0
8	.0025	.0042	.0010	.0001	.0402	.0134	0	0	.2681	0	0	0	.6703	0	0	0
7	.0005	.0029	.0041	.0004	0	.0402	.0134	0	0	.2681	0	0	0	.6703	0	0
6	.0005	.0002	.0027	.0040	.0004	0	.0402	.0134	0	0	.2681	0	0	0	.6703	0
5	.0004	.0003	0	.0027	.0040	.0004	0	.0402	.0134	0	0	.2681	0	0	0	.6703
4	.0001	.0004	.0002	0	.0027	.0040	.0004	0	.0402	.0134	0	0	.2681	0	0	.6703
3	0	0	.0004	.0002	0	.0027	.0041	.0003	0	.0436	.0101	0	.0670	.2011	0	.6703
2	0	0	0	.0004	.0002	0	.0035	.0035	.0002	.0101	.0369	.0067	.0670	.0670	.1341	.6703
1	0	0	0	.0002	.0004	.0001	.0015	.0035	.0102	.0102	.0168	.0235	.0704	.0670	.0670	.

Japan BMS 16 steps transition matrix

Japanese 16-step transition matrix

	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
16	.0071	.0108	.0115	.0128	.0508	.0328	.0265	.0272	.1594	.0465	.0362	.0348	.2709	.0259	.0235	.2231
15	.0043	.0112	.0137	.0131	.0104	.0635	.0360	.0274	.0140	.1827	.0452	.0348	.0287	.2682	.0235	.2231
14	.0043	.0044	.0164	.0169	.0108	.0085	.0791	.0390	.0143	.0131	.2058	.0439	.0287	.0259	.2657	.2231
13	.0042	.0045	.0041	.0231	.0164	.0090	.0073	.0970	.0280	.0133	.0119	.2287	.0377	.0259	.0235	.4654
12	.0032	.0052	.0043	.0035	.0275	.0169	.0078	.0057	.1032	.0292	.0121	.0106	.2467	.0350	.0235	.4654
11	.0015	.0048	.0060	.0039	.0029	.0313	.0174	.0064	.0047	.1117	.0280	.0108	.0529	.2198	.0326	.4654
10	.0015	.0014	.0063	.0065	.0034	.0024	.0348	.0173	.0054	.0107	.1129	.0267	.0531	.0502	.1931	.4745
9	.0014	.0015	.0012	.0074	.0069	.0030	.0029	.0365	.0173	.0115	.0144	.1091	.0689	.0504	.0477	.6199
8	.0009	.0017	.0014	.0011	.0084	.0072	.0037	.0039	.0372	.0235	.0150	.0179	.1463	.0641	.049	.6199
7	.0005	.0013	.0019	.0014	.0012	.0090	.0082	.0047	.0054	.0437	.0260	.0185	.0744	.1242	.0595	.6201
6	.0005	.0005	.0016	.0022	.0016	.0016	.0103	.0092	.0061	.0148	.0448	.0282	.0750	.0692	.1047	.6297
5	.0004	.0005	.0005	.0019	.0026	.0020	.0032	.0115	.0103	.0154	.0202	.0443	.0830	.0697	.0644	.6701
4	.0003	.0005	.0006	.0008	.0023	.0029	.0036	.0051	.0125	.0191	.0207	.0248	.0964	.0755	.0647	.6701
3	.0002	.0004	.0007	.0009	.0013	.0028	.0044	.0054	.0074	.0210	.0234	.0252	.0841	.0837	.0686	.6704
2	.0002	.0003	.0006	.0010	.0014	.0020	.0044	.0061	.0076	.0176	.0249	.0269	.0843	.0769	.0730	.6728
1	.0002	.0003	.0005	.0009	.0014	.0020	.0039	.0061	.0079	.0177	.0232	.0276	.0852	.0770	.0702	.6757

Japan BMS 128 steps transition matrix

Japanese 128-step transition matrix

	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
16	.0003	.0005	.0007	.0011	.0017	.0025	.0045	.0066	.0088	.0188	.0240	.0281	.0860	.0778	.0703	.6683
15	.0003	.0005	.0007	.0011	.0017	.0025	.0045	.0066	.0088	.0188	.0240	.0281	.0859	.0778	.0704	.6684
14	.0003	.0005	.0007	.0011	.0017	.0025	.0045	.0066	.0088	.0188	.0240	.0281	.0859	.0777	.0704	.6685
13	.0003	.0005	.0007	.0011	.0017	.0025	.0045	.0066	.0088	.0188	.0240	.0281	.0859	.0777	.0703	.6686
12	.0003	.0005	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0188	.0240	.0281	.0859	.0777	.0703	.6686
11	.0003	.0004	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0188	.0240	.0281	.0859	.0777	.0703	.6686
10	.0003	.0004	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0187	.0240	.0281	.0859	.0777	.0703	.6687
9	.0003	.0004	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0187	.0240	.0281	.0859	.0777	.0703	.6687
8	.0003	.0004	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0187	.0240	.0281	.0859	.0777	.0703	.6687
7	.0003	.0004	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0187	.0240	.0281	.0859	.0777	.0703	.6687
6	.0003	.0004	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0187	.0240	.0281	.0859	.0777	.0703	.6688
5	.0003	.0004	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0187	.0240	.0281	.0859	.0777	.0703	.6688
4	.0003	.0004	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0187	.0240	.0281	.0859	.0777	.0703	.6688
3	.0003	.0004	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0187	.0240	.0281	.0859	.0777	.0703	.6688
2	.0003	.0004	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0187	.0240	.0281	.0859	.0777	.0703	.6688
1	.0003	.0004	.0007	.0011	.0017	.0025	.0045	.0065	.0088	.0187	.0240	.0281	.0859	.0777	.0703	.6688

Discussion-Japan

- More than 81% of policyholders end up in classes 1, 2 or 3 and pay the minimum premium
- Less than .5% of policiholders end up in the malus zone. Why? Penalties are not severe enough.
- Income of insurer keeps decreasing

"The variability of premiums measure the degree of solidarity implied by the BMS"

Correct pricing of a safe driver

Advocate taking severity of claims into account in a BMS

- majority of optimal Bonus-Malus Systems (BMS) -assign to each policyholder a premium based on the number of his accidents only.
- same penalty for an accident of a small size or big size.
- optimal BMS designed are based both on the number of accidents of each policyholder and on the size of loss (severity) for each accident incurred.
- optimality is obtained by minimizing the insurers risk.

Literature Review

- Categorisation of Claim Severities (Discrete)
  - Picard(1976): Large and Small
  - Lemaire(1995): Property Damage and Bodily Injuries
  - Pitrebois et al., (2006): Four types, Dirichlet Distribution

- Distributions of Claim Severities (Continuous)
  - Frangos and Vrontos (2001): Pareto distribtution
  - Valdez and Frees (2005): Burr XII long-tailed distribution
  - Ni, Constantinescu, Pantelous (2014): Weibul distribution
  - Ni, Li, Constantinescu, Pantelous (2014): Weibul and Pareto distribution

Choice of distributions for claims severity

Tail behaviours of three comparative distributions [Boland 2007]

$$
\begin{aligned}
\textit{Exponential} : P(X > x) &= exp(-\theta x); \\
\textit{Weibull} : P(X > x) &= exp(-\theta x^{\gamma}); \\
\textit{Pareto} : P(X > x) &= \left(\frac{\theta}{\theta + x}\right)^{s}.
\end{aligned}
$$

Modelling Pareto Claim Severity

Mixing exponential with Inv.Gamma(m,s) results in Pareto(s,m)

$$F(x) = \int_0^{\infty} (1 - \mathrm{e}^{-\theta x}) \frac{e^{-m\theta}(\theta m)^{s+1}}{m\Gamma(s)} \, d\theta = 1 - \left(\frac{m}{m+x}\right)^s$$

Bayesian Approach - Posterior Distribution

$$\begin{aligned} \pi(\theta|\underbrace{x_1, x_2, \ldots, x_K}_{claims' \, history}) &= \frac{[\prod_{i=1}^{K} f(x_i|\theta)]\pi(\theta)}{\int_0^{\infty}[\prod_{i=1}^{K} f(x_i|\theta)]\pi(\theta)\mathrm{d}\theta} \\ &= \frac{e^{-\theta(m+M)}(\theta(m+M))^{K+s+1}}{(m+M)\Gamma(K+s)} \\ &\sim Inv.Gamma(s+K, m+M), \quad M = \sum_{k=1}^{K} x_k \end{aligned}$$

Premium Calculation

Mean Claim Severity

$$E[Severity] = \frac{m+M}{s+K-1}, \quad M = \sum_{k=1}^{K} x_k$$

The Net Premium

$$\textcolor{red}{Premium} = \underbrace{\frac{\theta+K}{t+\tau}}_{E(frequency)} \underbrace{\frac{m+M}{s+K-1}}_{E(severity)}$$

Modelling Weibull Claim Severity

Mixing exponential with a Levy distribution

$$F(x) = \int_0^\infty (1 - \mathrm{e}^{-\theta x}) \frac{c}{2\sqrt{\pi\theta^3}} exp\left(-\frac{c^2}{4\theta}\right) \mathrm{d}\theta = 1 - exp\left(-c\sqrt{x}\right).$$

Bayesian Approach

Posterior Distribution

$$\pi(\theta|x_1, x_2, \ldots, x_K) = \frac{[\prod_{i=1}^K f(x_i|\theta)]\pi(\theta)}{\int_0^\infty [\prod_{i=1}^K f(x_i|\theta)]\pi(\theta)\mathrm{d}\theta} = \frac{\theta^{K-\frac{2}{3}} exp\left(-\left(M\theta + \frac{c^2}{4\theta}\right)\right)}{\int_0^\infty \theta^{K-\frac{2}{3}} exp\left(-\left(M\theta + \frac{c^2}{4\theta}\right)\right) \mathrm{d}\theta}.$$

Using the modified Bessel Function

Bayesian Approach contd.

$$= \frac{\left(\frac{c}{2\sqrt{M}}\right)^{-\left(K-\frac{1}{2}\right)} \theta^{K-\frac{2}{3}} exp\left(-\left(M\theta + \frac{c^2}{4\theta}\right)\right)}{\int_0^\infty \left(\frac{2\sqrt{M\theta}}{c}\right)^{K-\frac{3}{2}} exp\left(-\frac{c\sqrt{M}}{2}\left(\frac{2\sqrt{M\theta}}{c} + \frac{c}{2\sqrt{M\theta}}\right)\right) \mathrm{d}\left(\frac{2\sqrt{M\theta}}{c}\right)}. \quad (1)$$

- The modified Bessel Function [Abramowitz and Stegun 1964]

$$B_v(x) = \int_0^\infty \mathrm{e}^{-x\cosh t}\cosh(vt)\mathrm{d}t.$$

- An alternative expression

$$B_v(x) = \frac{1}{2}\int_0^\infty exp\left(-\frac{1}{2}x\left(y + \frac{1}{y}\right)\right) y^{v-1}\mathrm{d}y, x > 0.$$

Posterior Distribution

Posterior Distribution: Generalised Inverse Gaussian (GIG)

$$\pi(\theta) = \frac{\left(\frac{\alpha'}{\beta'}\right)^{\frac{v}{2}} \theta^{v-1} exp\left(-\frac{1}{2}\left(\alpha'\theta + \frac{\beta'}{\theta}\right)\right)}{2B_v\left(\sqrt{\alpha'\beta'}\right)}.$$

where $\alpha' = 2M, \beta' = \frac{c^2}{2}, v = K - \frac{1}{2}$.
Best Estimate in the sense of using a quadratic loss funtion: posterior mean

$$E[GIG] = \sqrt{\frac{\beta'}{\alpha'}} \frac{B_{v+1}\left(\sqrt{\alpha'\beta'}\right)}{B_v\left(\sqrt{\alpha'\beta'}\right)}.$$

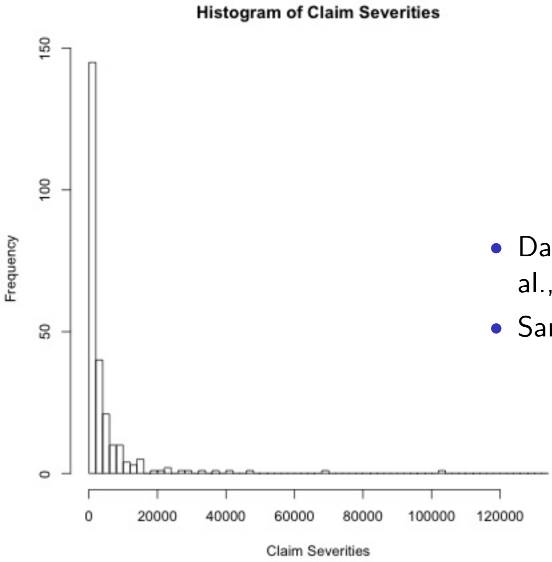
Mean Claim Severity

$$E[Severity] = \frac{1}{\theta_{t+1}(x_1, x_2, \ldots, x_K)} = \frac{2\sqrt{M}}{c} \frac{B_{K-\frac{1}{2}}(c\sqrt{M})}{B_{K+\frac{1}{2}}(c\sqrt{M})}. \quad (2)$$

Net Premium

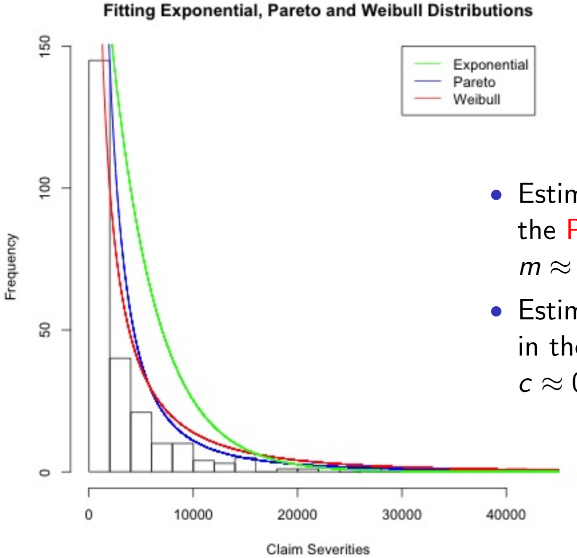
$$NetPremium = \begin{cases} \frac{\alpha+K}{t+\tau} \cdot \left(\frac{2\sqrt{M}}{c} \frac{B_{K-\frac{1}{2}}(c\sqrt{M})}{B_{K+\frac{1}{2}}(c\sqrt{M})} \right) & : \quad M > 0 \\ \left(\frac{\alpha}{t+\tau} \right) \left(\frac{2}{c^2} \right) & : \quad M = 0 \end{cases}$$

Numerical Illustration



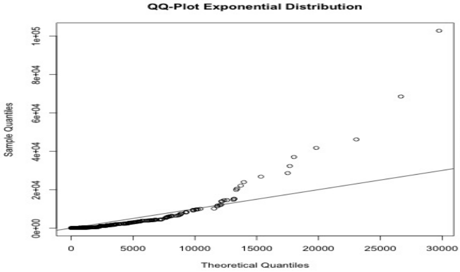
- Data source: [Klugman et al., 1998]
- Sample Size: 250

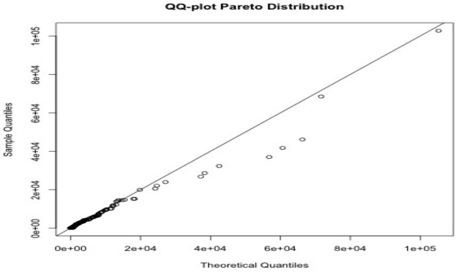
Fitting the Distributions (Maximum Likelihood Estimation)

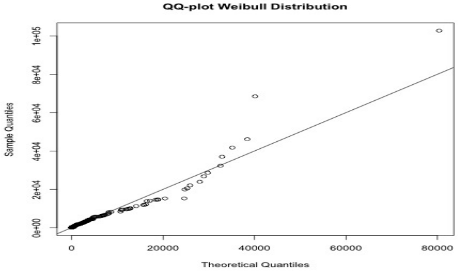


- Estimates of parameters in the Pareto distribution: $m \approx 2000; s \approx 1.34;$
- Estimates of the parameter in the Weibull distribution: $c \approx 0.02$

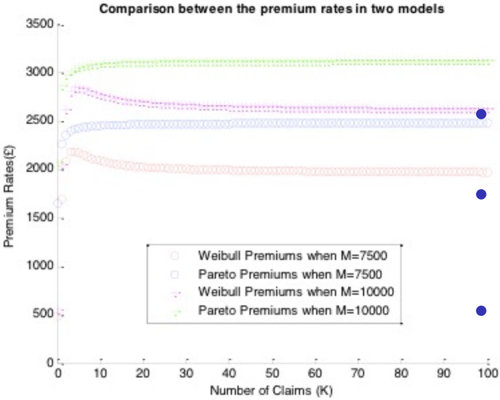
QQ Plots


QQ-Plot Exponential Distribution
Sample Quantiles
0e+00
2e+04
4e+04
6e+04
8e+04
1e+05
0
5000
10000
15000
20000
25000
30000
Theoretical Quantiles


QQ-plot Pareto Distribution
Sample Quantiles
0e+00
2e+04
4e+04
6e+04
8e+04
1e+05
0e+00
2e+04
4e+04
6e+04
8e+04
1e+05
Theoretical Quantiles


QQ-plot Weibull Distribution
Sample Quantiles
0e+00
2e+04
4e+04
6e+04
8e+04
1e+05
0
20000
40000
60000
80000
Theoretical Quantiles

Analysis of the behaviour



- The Weibull model offers cheaper premium rates;
- Up to a certain number of claims, the Weibull premium starts to slightly decrease;
- Bonus-Hunger issue.

Discussions

- The Weibul model is offering lower prices;
- The Weibull model is more applicable on the scenario where many small claims are filed;
- Weibull choice explained since reinsurance exists in practice to handle large claims;
- We might discourage the hunger for bonus phenomenon.

Mixed Strategy

- Weibull estimates better for small-sized claims.
- A mixture of the previous two models is suggested. With q denoting the probability that a claim cost exceeds a certain threshold z, we have

$$Premium = E_p[X_{wei}]E_p[N_{wei}](1-q) + E_p[X_{par}]E_p[N_{par}]q.$$

i.e., $X \sim X_{wei}$ when $X \leq z$ and $X \sim X_{par}$ when $X > z$. Note that q and z can both be observed from a sample and E_p stands for the posterior mean.

Frequency Distribution

- Suppose the total claim frequency is Negative Binomial distributed, i.e., $N \sim NB(\alpha, \tau)$. Then the number of claims above the limiting amount z also follows a Negative Binomial distribution, and $N_{par} \sim NB(\alpha, \tau q)$.
- Similarly, $N_{wei} \sim NB(\alpha, \tau(1-q))$.
- Apriori means of claim frequency (Pareto and Weibull claims)

$$
\begin{aligned}
E[N_{par}] &= \frac{\alpha}{\tau q}, \\
E[N_{wei}] &= \frac{\alpha}{\tau(1-q)}.
\end{aligned}
$$

- Bayesian posterior means

$$
\begin{aligned}
E_p[N_{par}] &= \frac{\alpha + qK}{\tau q + t}, \\
E_p[N_{wei}] &= \frac{\alpha + (1-q)K}{\tau(1-q) + t}.
\end{aligned}
$$

The Net Premium Formula

The Mixture Net Premium =

$$\frac{\alpha + K(1-q)}{\tau(1-q)+t} \cdot \frac{2\sqrt{M_1}}{c} \frac{B_{K(1-q)-\frac{1}{2}}(c\sqrt{M_1})}{B_{K(1-q)+\frac{1}{2}}(c\sqrt{M_1})}(1-q) + \frac{\alpha + Kq}{\tau q + t} \cdot \frac{m + M_2}{s + Kq - 1} q$$

- Kq or $K(1-q)$ are not necessarily integers.

Challenges of the future

Personalized premium based on telematics =driver behavior (examples)

Example 5:

Under the new rules, Philip and Jane, who have just obtained their driving licences, both pay 1000 euros a year for car insurance. However, if over time Jane appears to be a safer driver than Philip, her insurance premium will decrease more quickly than his based on her individual driving behaviour.

THANK YOU FOR YOUR ATTENTION!

HVALA!

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HAD

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